Effect of non-zero constant vorticity on the nonlinear resonances of capillary water waves

Adrian Constantin[†] and Elena Kartashova[*]
[†] *Faculty of Mathematics, University of Vienna, 1090 Vienna, Austria*
[*] *RISC, J. Kepler University, Linz 4040, Austria*

The influence of an underlying current on 3-wave interactions of capillary water waves is studied. The fact that in irrotational flow resonant 3-wave interactions are not possible can be invalidated by the presence of an underlying current of constant non-zero vorticity. We show that: 1) wave trains in flows with constant non-zero vorticity are possible only for two-dimensional flows; 2) only positive constant vorticities can trigger the appearance of three-wave resonances; 3) the number of positive constant vorticities which do trigger a resonance is countable; 4) the magnitude of a positive constant vorticity triggering a resonance can not be too small.

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1. Introduction. In this Letter we investigate the effect of a current on the dynamics of nonlinear capillary water waves, a problem of fundamental importance due to the ubiquity of currents at sea [8]. The most common force for creating water waves is the wind and wind generated capillary waves play a prominent role in the development of waves on water surfaces that are flat in the absence of wind. Indeed, capillary waves generate surface roughness allowing a better grip of the wind. This leads to the subsequent development of capillary-gravity and gravity waves as with increasing wave amplitude gravity becomes the dominant restoring force replacing surface tension [15, 18, 20]. In coastal navigation the important question arises whether the presence of an underlying current can be detected by investigating solely phenomena at the water's surface. Vorticity is adequate for the specification of a current. A uniform current is described by zero vorticity (irrotational flow), while the simplest example of a non-uniform current is that of tidal flows, which can be realistically modeled as two-dimensional flows with constant non-zero vorticity, with the sign of the vorticity distinguishing between ebb/tide [7]. Notice that in linear systems waves of different frequencies do not interact due to the superposition principle, while in a nonlinear system the lowest-order nonlinear effect (with respect to an expansion in the wave amplitudes) is the resonant interaction of three waves of different wave-vectors and frequencies. Resonant interactions can profoundly affect the evolution of waves by making significant energy transfer possible among the dominant wave trains, accounting thus for wave patterns that are higher and steeper than linear wave theory would predict and providing insight into the effects of weak turbulence [19]. It is known that capillary waves in irrotational flow do not exhibit exact 3-wave resonance [11]. We will show that the presence of an underlying current of constant vorticity can lead to the excitation of nonlinear resonances, but only for special vorticities.

2. Dispersion function. Our purpose is to show that currents strongly affect the dynamics already at the level of capillary waves. To emphasize the wave-current interactions we consider a setting with a flat bed in order to minimize the effect of the shape of the shoreline and of the water bed on such flows. In this context we investigate whether or not three-wave resonances are possible among capillary waves. The dispersion relation for exact solutions to the governing equations for capillary-gravity water waves propagating at the free surface of water of constant density $\rho = 1$ with a flat bed and in a flow with constant vorticity Ω is

$$\begin{aligned}\omega &= \frac{\Omega}{2}\tanh(kd)\\ &+ \frac{1}{2}\sqrt{\Omega^2\tanh^2(kd)+4(kg+k^3\sigma)\tanh(kd)},\end{aligned} \quad (1)$$

where $d > 0$ is the average water depth, g is the gravitational constant of acceleration and $\sigma > 0$ is the coefficient of surface tension, and the notation k is used for the modulus of the wave vector, $k = |\mathbf{k}|$. The case of capillary waves corresponds to $g \to 0$ and that of gravity waves to $\sigma \to 0$ cf. [4, 22, 23]. Moreover, capillary waves being short waves, it is appropriate to take the short wave limit $kd \to \infty$ of the above dispersion relation; as $\tanh(kd) \to 1$ we obtain the dispersion relation for small-amplitude capillary waves as

$$\omega = \frac{\Omega}{2} + \frac{1}{2}\sqrt{\Omega^2 + 4k^3\sigma}. \quad (2)$$

In the case of irrotational flows (with zero vorticity), the dispersion relation reads [23]

$$\omega^2 = \sigma k^3. \quad (3)$$

As it was shown in [11], exact three-wave resonances are prohibited for (3) by purely kinematic considerations. Below we show that in the case of non-zero constant vorticity, resonances are possible but only for a countable number of positive vorticities.

3. Magnitudes of vorticity. First, we show that the resonance conditions

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)=\omega(\mathbf{k}_3), \quad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3 \tag{4}$$

do have solutions for the dispersion function (2). Let us introduce the variable $\zeta=\Omega/2\sqrt{\sigma}$ and rewrite (4) as

$$\sqrt{\zeta^2+k_1^3}+\sqrt{\zeta^2+k_2^3}=\sqrt{\zeta^2+k_3^3}-\zeta, \quad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3.$$

Our primary goal is investigating whether for a given current strength Ω three-wave resonances are possible for wave vectors with integer coordinates. Since capillary wave trains propagating in flows of constant vorticity are only possible if the flow is two-dimensional (see Section 5), only the case of one-dimensional wave-vectors, $\dim \mathbf{k}=1$, is relevant. Now $\mathbf{k}=k$ and (4) reads

$$\sqrt{\zeta^2+k_1^3}+\sqrt{\zeta^2+k_2^3}=\sqrt{\zeta^2+k_3^3}-\zeta, \tag{5}$$

$$k_1+k_2=k_3, \tag{6}$$

and we will study the magnitudes of vorticities corresponding to integer wave-vectors $k_1,\ k_2,\ k_3>0$. To investigate (5),(6) it is appropriate to view it as an equation in the unknown $\zeta\neq 0$, with $k_1,\ k_2,\ k_3>0$ given integers. Let us first simplify notation by denoting

$$k_1^3=\alpha, \quad k_2^3=\beta, \quad k_3^3=\gamma, \quad E=\alpha+\beta-\gamma. \tag{7}$$

Since irrespective of the sign of ζ both sides of (5) are positive, squaring them we see that (5) is equivalent to

$$E+2\zeta\sqrt{\zeta^2+\gamma}=-2\sqrt{(\zeta^2+\alpha)(\zeta^2+\beta)}. \tag{8}$$

Once we obtain a formula for ζ^2, equation (8) yields

$$\zeta=-\frac{E+2\sqrt{(\zeta^2+\alpha)(\zeta^2+\beta)}}{2\sqrt{\zeta^2+\gamma}}. \tag{9}$$

But (8) squared yields

$$E^2-4\zeta^2E-4\alpha\beta=-4E\zeta\sqrt{\zeta^2+\gamma},$$

which, squared again, leads to

$$(E^2-4\alpha\beta)^2=$$
$$8E\,\zeta\,(2E\gamma+E^2-4\alpha\beta)=8E\,\zeta^2\,[(\alpha-\beta)^2-\gamma^2],$$

with the last equality a simple consequence of the definition of E. Thus

$$\zeta^2=\frac{(E^2-4\alpha\beta)^2}{8\,(\alpha+\beta-\gamma)[(\alpha-\beta)^2-\gamma^2]}. \tag{10}$$

Obviously, (10) can be regarded as a 2-parameter series of solutions for (5)-(6). Indeed, for any wave-vectors k_1,k_2 there are at most two vorticities Ω generating a three-wave resonance of the form (5)-(6). For instance, $k_1=k_2=1$ yield $k_3=2,\ \alpha=\beta=1,\ \gamma=8,\ E=-6,\ \zeta^2=1/3$ and the corresponding vorticity Ω has to satisfy $\Omega^2=4\sigma/3$. Now we have to gain more understanding about the possible sign of the vorticity. To formulate our theorem on the sign of vorticity, let us first introduce a new variable

$$\delta=\sqrt{k_3^3}-\sqrt{k_1^3}-\sqrt{k_2^3}. \tag{11}$$

Theorem 1 (on the sign on vorticity). *Three wave-vectors $k_1,\ k_2,\ k_3\geq 1$ are solutions of (5),(6) if and only if $\delta>0$, in which case the vorticity $\Omega=2\sqrt{\sigma}\zeta$ is positive, with ζ given by (9).*

Proof. Two observations can be made immediately. Firstly, as it was shown in [11], the equality $\delta=0$ can be transformed to the particular cubic case of Fermat's Last Theorem and therefore has no integer solutions. Secondly, condition (6) yields

$$k_3^3>k_1^3+k_2^3 \tag{12}$$

and therefore $E<0$.

We first claim that if $k_1,\ k_2,\ k_3\geq 1$ are solutions of (5),(6), then $\delta>0$. Indeed, notice that the inequality

$$3k_1^2k_2+3k_1k_2^2=3k_1k_2(k_1+k_2)\geq 6k_1k_2\sqrt{k_1k_2}$$

yields $3k_1^2k_2+3k_1k_2^2>2\sqrt{k_1^3k_2^3}$. If (6) holds, adding $k_1^3+k_2^3$ to both sides of the above inequality, we get

$$k_3^3\geq\left(\sqrt{k_1^3}+\sqrt{k_2^3}\right)^2,$$

that is, $\delta>0$.

Let us now prove that for $\delta>0$ we always have a solution $\zeta>0$ and no solution $\zeta<0$. The statement about the positive solution follows at once by noticing that the function $\zeta\mapsto\sqrt{\zeta^2+\alpha}+\sqrt{\zeta^2+\beta}-\sqrt{\zeta^2+\gamma}+\zeta$ has limit $+\infty$ for $y\to\infty$, being strictly increasing on $(0,\infty)$ since its derivative satisfies

$$\frac{\zeta}{\sqrt{\zeta^2+\alpha}}+\frac{\zeta}{\sqrt{\zeta^2+\beta}}-\frac{\zeta}{\sqrt{\zeta^2+\gamma}}+1>\frac{\zeta}{\sqrt{\zeta^2+\alpha}}+1>1,$$

for $\zeta>0$, as $\gamma>\beta$ in view of (12). Therefore this function has a zero on $(0,\infty)$ if and only if its value at $\zeta=0$ is strictly negative, that is, if and only if $\delta>0$. Now notice that we know by (9) and (10) that if a solution exists, it is unique. This concludes the proof.

Corollary 1 *The set of all resonant triads covered by (5),(6), is generated by a countable number of positive vorticities.*

More precisely, given the wave-vectors $k_1,\ k_2\geq 1$ we define k_3 via (6), and compute

$$E=-3k_1k_2\,(k_1+k_2),$$
$$E^2-4\alpha\beta=k_1^2k_2^2\,(9k_1^2+9k_2^2+14k_1k_2),$$

finding also that $\gamma^2 - (\alpha-\beta)^2$ equals

$$k_1 k_2\,(6k_1^4 + 15k_1^3 k_2 + 22k_1^2 k_2^2 + 15k_1 k_2^3 + 6k_2^4).$$

Knowing that $\zeta > 0$, we infer from (10) that a three-wave resonance occurs only if Ω is given explicitly by

$$\frac{\sqrt{\sigma}\,k_1^{3/2}k_2^{3/2}\,(9k_1^2+9k_2^2+14k_1k_2)}{\sqrt{6(k_1+k_2)(6k_1^4+15k_1^3k_2+22k_1^2k_2^2+15k_1k_2^3+6k_2^4)}}. \tag{13}$$

For $k_1 \geq k_2$ we see that the denominator is bounded from above by $16\sqrt{3}\,k_1^{5/2}$ and from below by $16\sqrt{3}\,k_2^{5/2}$. Since

$$9k_1^2 + 9k_2^2 + 14k_1k_2 = 9(k_1-k_2)^2 + 32k_1k_2 \geq 32k_1k_2,$$

a lower bound for the numerator is $32\sqrt{\sigma}\,k_1^{5/2}k_2^{5/2}$, with the evident upper bound $32\sqrt{\sigma}\,k_1^{7/2}k_2^{3/2}$. Therefore

$$2\sqrt{\frac{\sigma}{3}}\,k_1^{5/2}\,\frac{k_1}{k_2} \geq \Omega \geq 2\sqrt{\frac{\sigma}{3}}\,k_2^{5/2}.$$

Corollary 2 *Three-wave resonant interactions do not occur in flows with sufficiently small constant vorticity, the minimal magnitude of a resonance generating vorticity being* $\Omega_{min} = 2\sqrt{\sigma}/\sqrt{3}$.

4. Structure of resonances. In the previous section we have shown that for any two wave-vectors k_1, k_2, a corresponding resonant triad (k_1, k_2, k_1+k_2) will be generated by the vorticity given by (13). The inverse problem — to compute all resonant triads generated by a given vorticity — is much more complicated. Already for one fixed wave-vector, this problem is equivalent to finding rational points on an elliptic curve. On the other hand, it is relatively easy to compute numerically, in a some finite spectral domain, the set of all resonant triads and the corresponding magnitudes of the vorticity. Another reasonable question is whether different resonant triads can be generated by almost the same magnitude of vorticity, with some accuracy ϵ because these type of resonances might enrich the cluster structure.

The structure of resonance clusters has been investigated numerically, in the spectral domain $k_1, k_2 \leq 100$. All exact solutions for ζ^2, that is, solutions with $\epsilon = 0$ have one of two forms:

1. Each resonance triad of the form $(k_1, k_1, 2k_1)$ with $k_1 = 1, 2, ..., 100$ can be generated by only one vorticity; two different triads of this form, $(k_1, k_1, 2k_1)$ and $(k_2, k_2, 2k_2)$ with $k_1 \neq k_2$ can only be generated by two different vorticities.

2. Each resonant triad of the general form $(k_1, k_2, k_1 + k_2)$ with $k_1 \neq k_2$ has the corresponding symmetrical resonant triad $(k_2, k_1, k_1 + k_2)$. Both symmetrical triads are generated by the same vorticity. This fact can be observed immediately from the form of (13) which is invariant under the transformation $[k_1 \mapsto k_2, \quad k_2 \mapsto k_1]$.

It appears that a given vorticity can generate not more than two exact resonant triads. The situation does not change substantially when $\epsilon \neq 0$: again, not more than two resonant triads appear to be generated by almost the same magnitude of vorticity. For example,

$$\begin{aligned}
\epsilon &= 10^{-1} \Rightarrow (1,9,10), && \zeta^2 = 4.824;\\
\epsilon &= 10^{-1} \Rightarrow (2,3,5), && \zeta^2 = 4.764;\\
\epsilon &= 10^{-2} \Rightarrow (2,14,16), && \zeta^2 = 29.5622;\\
\epsilon &= 10^{-2} \Rightarrow (4,5,9), && \zeta^2 = 29.5612;\\
\epsilon &= 10^{-4} \Rightarrow (2,88,90), && \zeta^2 = 196.212144;\\
\epsilon &= 10^{-4} \Rightarrow (3,40,43), && \zeta^2 = 196.212121;
\end{aligned}$$

The only difference with the case $\epsilon = 0$ is that now the two co-existing triads are not symmetrical anymore. In Table I the distribution of the resonance generating vorticities is given. One can see immediately that more that 30% of all vorticities are relatively small, in the sense that they satisfy the condition $\Omega = 2\zeta\sqrt{\sigma} < 2\cdot 10^2\sqrt{\sigma}$. (first row on the left in the Table I). The number of resonance generating vorticities in the partial domains is decreasing exponentially with the growth of the general spectral domain. Interestingly enough, new asymmetrical triads appear for ϵ of the order of 10^{-4} or bigger, i.e. ϵ is bounded from below (cf. detuned frequency balance in the irrotational case [12, 21]).

Magnitude of $\tilde{\zeta}$	Number of Ω	Magnitude of $\tilde{\zeta}$	Number of Ω
$\tilde{\zeta} < 1$	3477	$4 \leq \tilde{\zeta} < 5$	494
$1 \leq \tilde{\zeta} < 2$	1146	$5 \leq \tilde{\zeta} < 6$	407
$2 \leq \tilde{\zeta} < 3$	781	$6 \leq \tilde{\zeta} < 7$	363
$3 \leq \tilde{\zeta} < 4$	609	$7 \leq \tilde{\zeta}$	2723

TABLE I: The distribution of the magnitudes of the resonance generating vorticities is presented, in the spectral domain $k_1, k_2 \leq 100$. The notation $\tilde{\zeta} = \zeta^2 \cdot 10^4$ is used.

Generically, to describe the dynamics of the resonances in the rotational case, two steps have to be performed. Firstly, nonlinear evolution equations governing the rotational case should be derived. Afterwards the standard procedure would be to construct all resonant clusters met in the solution set above and write out explicitly the corresponding dynamical systems [13]. Secondly, the interaction coefficient has to be computed which, of course, will be different from the coefficient known for irrotational flows. Computing the explicit form of the interaction coefficient for a given wave system is a non-trivial technical problem demanding tedious computations which can not be automatized in the present state of symbolic programming [14].

5. Flows with constant vorticity. Let us recall the governing equations for capillary water waves propagating at the free surface of a layer of water above a flat

bed [9]. In the fluid domain of average depth $d > 0$ bounded above by the free surface $z = d + \eta(x, y, t)$ and below by the flat bed $z = 0$, the velocity field $\mathbf{u}(\mathbf{x}, t)$ and the pressure function $P(\mathbf{x}, t)$, where $\mathbf{x} = (x, y, z)$ and $\mathbf{u} = (u_1, u_2, u_3)$, satisfy the Euler equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla P = 0, \tag{14}$$

as well as the equation of mass conservation

$$\nabla \cdot \mathbf{u} = 0, \tag{15}$$

expressing homogeneity (with constant water density $\rho = 1$). These equations are coupled with the boundary conditions

$$P = \sigma H, \quad u_3 = \eta_t + u_1\,\eta_x + u_2\,\eta_y, \tag{16}$$

on the free surface $z = d + \eta(x, y, t)$, the constant $\sigma > 0$ being the surface tension coefficient and H being twice the mean curvature,

$$H(x, y, t) = \frac{(1 + \eta_x^2)\eta_{yy} - 2\eta_x\eta_y\eta_{xy} + (1 + \eta_y^2)\eta_{xx}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}}.$$

On the flat bed $z = 0$ we require

$$u_3 = 0. \tag{17}$$

The governing equations for capillary waves are (14)-(17). These equations are well-posed [6]: starting with an initial surface profile $\eta(x, y, 0)$ with Sobolev regularity $H^{6.5}$ and an initial velocity field $\mathbf{u}(\mathbf{x}, 0)$ with Sobolev regularity $H^{5.5}$ satisfying (15), for some time $t \in [0, T)$ with $T > 0$ there exists a unique solution $(\mathbf{u}, \eta, P)$ of the system (14)-(17). In our context it is important to keep track of the vorticity $\Omega(\mathbf{x}, t)$, obtained at any time t from the velocity profile as $\nabla \times \mathbf{u} = \Omega$. With this purpose, notice that using (14)-(17) one can derive the vorticity equation [16]

$$\Omega_t + (\mathbf{u} \cdot \nabla)\,\Omega = (\Omega \cdot \nabla)\,\mathbf{u}. \tag{18}$$

To investigate further the vorticity, it is useful to introduce the flow map $\mathbf{x} \mapsto \Phi(\mathbf{x}, t)$: this map advances each particle in the water region from its position $\mathbf{x}$ at time $t = 0$ to its position $\Phi(\mathbf{x}, t)$ at time t. For fixed t, Φ is an invertible smooth mapping and from (14), (15), (18) one can infer that [16]

$$\Omega(\Phi(\mathbf{x}, t), t) = J(\mathbf{x}, t)\,\Omega(\mathbf{x}, 0), \tag{19}$$

where $J(\mathbf{x}, t)$ is the Jacobian matrix of the flow map. An immediate consequence of (19) is that in three-dimensional flows a particle which has no vorticity never acquires it and conversely, a particle which is moving rotationally will continue to do so.

Theorem 2 (on the dimension of flows with constant vorticity). *Capillary wave trains can propagate at the free surface of a layer of water with a flat bed in a flow of constant non-zero vorticity only if the flow is two-dimensional.*

Proof. A wave train is a periodic surface wave which propagates without change of shape at constant speed $c > 0$ in a fixed direction, say, that of the x-coordinate, and which is unchanged in the y-direction (horizontal and orthogonal to the wave propagation direction). Notice that for a two-dimensional water flow (in our setting, independent of the y-coordinate), we have $\Omega_1 = \Omega_3 = 0$ so that $(\Omega \cdot \nabla)\,\mathbf{u} = \Omega_2\,\mathbf{u}_y = 0$ and therefore (18) implies that $\Omega_t + (\mathbf{u} \cdot \nabla)\,\Omega = 0$: the vorticity of each individual water particle is conserved as the particle moves about. In particular, if initially the vorticity is constant, it will stay so. The existence of wave trains is ensured in this setting provided the wave speed is given by the dispersion relation (1) cf. [24].

Conversely, consider the wave train $z = d + \eta(x - ct)$ propagating in a flow of constant vorticity $\Omega \neq 0$. Then $(\Omega \cdot \nabla)\,\mathbf{u} = 0$ by (18), that is, at every instant the vector $\mathbf{u}$ is constant in the direction of Ω. We first claim that for non-flat free surfaces, the direction Ω has to be horizontal. Indeed, if $\Omega_3 \neq 0$, this in combination with (17) would yield that $u_3 \equiv 0$ throughout the flow. The third component of (14) then forces $P_z = 0$ so that

$$P(x, y, z, t) = \sigma H(x - ct) \tag{20}$$

throughout the flow in view of the first equation in (16). Moreover, $\partial_z u_2 = -\Omega_1$ and $\partial_z u_1 = \Omega_2$ yield

$$\begin{cases} u_1(x, y, z, t) = v_1(x, y, t) + \Omega_2 z, \\ u_2(x, y, z, t) = v_2(x, y, t) - \Omega_1 z. \end{cases} \tag{21}$$

Further, from (15) we infer the existence of a function $\psi(x, y, t)$ satisfying

$$\psi_x = -v_2, \qquad \psi_y = v_1. \tag{22}$$

Writing the first two components of (14) on the flat bed $z = 0$ and on $z = \varepsilon > 0$ with $\varepsilon > 0$ small enough for this horizontal plane to be in the fluid domain, we get

$$\Omega_2\psi_{xy} - \Omega_1\psi_{yy} = -\Omega_2\psi_{xx} + \Omega_1\psi_{xy} = 0.$$

As $\Omega_3 = \partial_x u_2 - \partial_y u_1$ yields

$$(\partial_x^2 + \partial_y^2)\psi = -\Omega_3, \tag{23}$$

and $(\Omega \cdot \nabla)\,\mathbf{u} = 0$ yields

$$\Omega_1\psi_{xy} + \Omega_2\psi_{yy} + \Omega_2\Omega_3 = -\Omega_1\psi_{xx} - \Omega_2\psi_{xy} - \Omega_1\Omega_3 = 0,$$

unless $\Omega_1 = \Omega_2 = 0$ we infer from these relations that

$$\psi_{xx} = -\frac{\Omega_1^2\Omega_3}{\Omega_1^2 + \Omega_2^2}, \ \psi_{xy} = -\frac{\Omega_1\Omega_2\Omega_3}{\Omega_1^2 + \Omega_2^2}, \ \psi_{yy} = -\frac{\Omega_2^2\Omega_3}{\Omega_1^2 + \Omega_2^2}.$$

Therefore, assuming $\Omega_1^2+\Omega_2^2>0$, we get

$$\psi(t,x,y)=A\,y^2+B\,xy+C\,x^2+a(t)\,x+b(t)\,y+k(t),$$

for some functions a, b, k, where we denoted

$$A=-\frac{\Omega_2^2\Omega_3}{2(\Omega_1^2+\Omega_2^2)},\ B=-\frac{\Omega_1\Omega_2\Omega_3}{\Omega_1^2+\Omega_2^2},\ C=-\frac{\Omega_1^2\Omega_3}{2(\Omega_1^2+\Omega_2^2)}.$$

From (14) we now infer that

$$P_x=-b'(t)-B\,b(t)+2A\,a(t),\ P_y=a'(t)+2C\,b(t)-B\,a(t),$$

and (20) then yields

$$P(x,y,t)=\alpha\,(x-ct)+\beta$$

for some constants α, β. But then $P=\sigma H$ is impossible since the right-hand side has by periodicity infinitely many zeros (at least one between two consecutive crests, by the mean-value theorem). To rule out the possibility that $\Omega_3\neq 0$, it remains to consider the case $\Omega_1=\Omega_2=0$. If this holds, then the functions u_1 and u_2 are independent of z in view of (21) and are harmonic in (x,y) by (22)-(23). The second equation in (16) yields $[u_1(x,y,t)-c]\,\eta_x(x-ct)=0$ for all x, y real. But the function u_1 is a harmonic function of (x,y) and taking into account the structure of the level sets of a harmonic function (they are curves in the plane unless the function is constant cf. [2]), this yields $u_1\equiv c$ since $\eta_x\not\equiv 0$. But then the first equation in (14) ensures $P_x=0$, while (15) yields that u_2 is independent of y. As $\Omega_3=\partial_x u_2-\partial_y u_1$ we have $u_2(t,x,y)=\Omega_3 x+f(t)$ for some function f. Now the second equation in (14) yields $P_y=-f'(t)-c\Omega_3$. But this can be reconciled with $P(x,y,t)=\sigma H(x-ct)$ only if H is a constant and then the surface has to be flat [1, 17].

We therefore proved that $\Omega_3=0$. It remains to show that $\Omega_1=0$ and that P and $\mathbf{u}$ are independent of y. Since $\Omega_1\partial_x u_i+\Omega_2\partial_y u_i=0$ for $i=1,2,3$ and $\nabla\times\mathbf{u}=\Omega$ throughout the flow, we first infer that

$$\Omega_1 u_1+\Omega_2 u_2=h(t) \tag{24}$$

for some function h, since all spatial derivatives of the left-hand side are zero. Multiplying the first equation in (14) by Ω_1, the second equation by Ω_2, and adding up, we infer from the above relations that

$$\Omega_1 P_x+\Omega_2 P_y=-h'(t) \tag{25}$$

throughout the fluid. Notice that once we show that $\Omega_1=0$, then $\Omega_2\neq 0$ as $\Omega\neq 0$, and $(\Omega\cdot\nabla)\mathbf{u}=0$ ensures that $\mathbf{u}$ does not depend on y. Moreover, in this case (24) yields that u_2 is simply a function of time and consequently the second equation in (14) forces P to be independent of y. To complete the proof it therefore suffices to show that $\Omega_1=0$. Assuming $\Omega_1\neq 0$, the relation $\partial_x u_2-\partial_y u_1=0$ ensures the existence of a function $\varphi(\mathbf{x},t)$ satisfying

$$u_1=\varphi_x,\quad u_2=\varphi_y.$$

From $\nabla\times\mathbf{u}=\Omega$ we infer

$$\partial_y u_3=\varphi_{yz}+\Omega_1,\quad \partial_x u_3=\varphi_{xz}-\Omega_2,$$

so that, adding if necessary a function depending only on (z,t) to φ, we have

$$u_3=\varphi_z+\Omega_1 y-\Omega_2 x.$$

Denoting $\xi=\Omega_2 x-\Omega_1 y$, we get

$$u_1=\varphi_x,\quad u_2=\varphi_y,\quad u_3=\varphi_z-\xi, \tag{26}$$

and (15) becomes

$$\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0. \tag{27}$$

From (24) and (25) we deduce that φ and P admit the decomposition

$$\begin{cases}\varphi(\mathbf{x},t) &= \ \ (\mu x+\nu y)\,h(t)+F(\xi,z,t),\\ P(\mathbf{x},t) &= -(\mu x+\nu y)\,h'(t)+G(\xi,z,t),\end{cases} \tag{28}$$

for some functions F, G, where we denoted

$$\mu=\frac{\Omega_1}{\Omega_1^2+\Omega_2^2},\ \nu=\frac{\Omega_2}{\Omega_1^2+\Omega_2^2}.$$

Using (26) and (28), we can express (14) as

$$\begin{cases}\Omega_2\,\partial_\xi\,[F_t+\frac{\Omega_1^2+\Omega_2^2}{2}\,F_\xi^2+\frac12\,F_z^2-\xi F_z+G]=0,\\ \Omega_1\,\partial_\xi\,[F_t+\frac{\Omega_1^2+\Omega_2^2}{2}\,F_\xi^2+\frac12\,F_z^2-\xi F_z+G]=0,\\ \partial_z\,[F_t+\frac{\Omega_1^2+\Omega_2^2}{2}\,F_\xi^2+\frac12\,F_z^2-\xi F_z+G]=(\Omega_1^2+\Omega_2^2)\,F_\xi.\end{cases}$$

Under our working assumption $\Omega_1\neq 0$ these relations imply $F_{\xi\xi}=0$. But since (27) and (28) yield

$$F_{zz}+(\Omega_1^2+\Omega_2^2)\,F_{\xi\xi}=0,$$

we also have $F_{zz}=0$. Consequently

$$F(\xi,z,t)=f_0(t)\,\xi z+f_1(t)\,\xi+f_2(t)\,z+f_3(t)$$

for some functions f_i, $i=0,1,2,3$. From (26) and (28) we now infer that

$$\begin{cases}u_1(\mathbf{x},t) &= \ -\Omega_2\,[f_0(t)\,z+f_1(t)]+\mu\,h(t),\\ u_2(\mathbf{x},t) &= \ \ \Omega_1\,[f_0(t)\,z+f_1(t)]+\nu\,h(t),\\ u_3(\mathbf{x},t) &= [f_0(t)-1]\,(\Omega_2 x-\Omega_1 y)+f_2(t).\end{cases}$$

But then equating the coefficient of y on both sides of the second relation in (16), stating that

$$u_3=(u_1-c)\,\eta_x(x-ct)\quad\text{on}\quad z=d+\eta(x-ct), \tag{29}$$

we first infer that $f_0(t)\equiv 1$ as $\Omega_1\neq 0$ by assumption. But then $u_3(\mathbf{x},t)=f_2(t)$ and now the right-hand side of (29) is periodic in x and the left-hand side is independent

of x. Therefore $\eta_x \equiv 0$ which contradicts the fact that the free surface was not flat. This concludes the proof.

Theorem 2 allows us to consider for flows of constant vorticity only two-dimensional flows propagating in the x-direction, for which the vorticity vector takes the form $(0, \Omega_2, 0)$. For such flows it is therefore natural by abuse of notation to identify the vorticity vector Ω with its second component. For irrotational flows $\Omega = 0$ so that a velocity potential exists and methods from harmonic function theory (involving the Dirichlet-Neumann operator) can be used to transform the governing equations for capillary water waves to a Hamiltonian system expressed solely in terms of the free surface $\eta(x,t)$ and of the restriction of the velocity potential on the free surface [25]. The absence of a velocity potential for non-zero vorticities complicates the analysis considerably cf. [5].

6. Discussion. Our main conclusions can be formulated as follows:

- Flows with constant non-zero vorticity admit capillary wave trains only if they are two-dimensional.
- Only positive vorticity can trigger the appearance of three-wave resonances.
- The number of positive vorticities which do trigger a resonance is countable.
- The magnitude of a positive vorticity triggering a resonance can not be too small.

It is remarkable that relatively large positive constant vorticities are necessary to observe resonant 3-wave interactions. The fact that resonances can never occur for negative constant vorticities substantiates the belief that the vorticity of the flow has a big influence on the dynamics of the surface water waves. In this context the “frozen turbulence” observed in numerical simulations [19] for capillary waves is just a manifestation of the fact that resonances are absent.

To understand the nonlinear resonance dynamics among rotational capillary waves, an evolution equation corresponding to the rotational case should be derived. The presence of non-zero vorticity invalidates the existence of a velocity potential for the flow, as is the case for irrotational flows, and harmonic function theory is not readily available for the analysis. Two common methods of analysis to find the exact form of the dynamic equations are the method of multiple scales and variational techniques [10]. Both methods are applicable for flows with vorticity [3, 5, 9, 24]; this is work in progress.

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* Electronic address: `adrian.constantin@univie.ac.at`, `lena@risc.uni-linz.ac.at`

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